Proof of Pohlke's Theorem and its Generalizations by Affinity.

By Arnold Emch.

1. *Introduction.*

The purpose of this paper is to show how Pohlke's Theorem, its generalizations, and some related propositions may be proved in a comprehensive manner by making use of affine collineations in space.

The theorem was first published without proof in the first part of Pohlke's "Descriptive Geometry" in 1860, and may be stated as follows:

Three straight line segments of arbitrary length in a plane, drawn from a point and making arbitrary angles with each other, form a parallel projection of three equal segments drawn from the origin on three rectangular coordinate-axes; however, only one of the segments, or one of the angles can vanish.

The first elementary rigorous proof of the fundamental theorem of axonometry, as Pohlke's Theorem is sometimes called, was given by H. A. Schwarz.* Subsequently numerous other proofs of the theorem and, in a few instances, of its generalization for an oblique system of coordinate-axes were given.†

* *Crelle's Journal,* Vol. LXIII (1864), pp. 309–314, "Elementarer Beweis des Pohlkeschen Fundamentalsatzes der Axonometrie.

† von Deschwanden, who received his knowledge of the theorem from Steiner on one of the latter's visits to Zürich, gave a proof in the *Vierteljahrsschrift of the Maturforschende Gesellschaft in Zürich,* Vol. VI (1861), pp. 254–284, which, however, was not entirely satisfactory.

In the same volume, pp. 358–367, Kinkelin gave an analytic proof.

In Vol. XI, pp. 350–358, of the same publication, Reye, by means of projective geometry, generalized the theorem for oblique coordinates.

Among others who gave purely geometric demonstrations of the theorem, and constructive solutions of the problem involved may be mentioned:

Pelz, *Wiener Berichte,* Vol. LXXVI, II (1877), pp. 123–128.

Peschka, *Ibid.,* Vol. LXXVIII, II (1879), pp. 1043–1055.

Mandel, *Ibid.,* Vol. XCIV, II (1886), pp. 60–65.

Ruth, *Ibid.,* Vol. C, II (1891), pp. 1088–1092.

Schur, *Mathematische Annalen,* Vol. XXV (1885), pp. 596–597.

Schur, *Crelle's Journal,* Vol. CXVII (1896), pp. 474–475.

Küpper, *Mathematische Annalen,* Vol. XXXIII (1889), pp. 474–475.

Beck, *Crelle's Journal,* Vol. CVI (1890), pp. 121–124.

Schilling, *Zeitschrift für Mathematik und Physik,* Vol. XLVIII (1903), pp. 487–494.

Loria, *Vorlesungen über darstellende Geometrie,* Vol. I (1907), pp. 190–194,

Grossmann, *Darstellende Geometrie* (1915), pp. 26–29,

and various other well-known texts on descriptive geometry make use of the theorem in the discussion of axonometry.

I shall first investigate some of the properties of affine collineations in space, as far as they are related to the problem involved. Based upon these properties it will then not be difficult to prove Pohlke's and a number of similar theorems.

2. *Definition and General Properties of Affinity.*

Let OX, OY, OZ and $O'X'$, $O'Y'$, $O'Z'$ be two systems of coordinates, which, for the sake of definiteness, we assume as orthogonal; then the two spaces are defined as related by affinity when their coordinates are connected by the substitution

$$S \equiv \left\{ \begin{array}{l} x' = a_0 + a_1 x + a_2 y + a_3 z, \\ y' = b_0 + b_1 x + b_2 y + b_3 z, \\ z' = c_0 + c_1 x + c_2 y + c_3 z. \end{array} \right\} \tag{1}$$

The classification * of affinities depends upon the properties of the matrix

$$\left\| \begin{array}{cccc} a_1 - 1 & a_2 & a_3 & a_0 \\ b_1 & b_2 - 1 & b_3 & b_0 \\ c_1 & c_2 & c_3 - 1 & c_0 \end{array} \right\|. \tag{2}$$

They form a projective twelve-parameter group and leave the plane at infinity invariant. Parallel planes and parallel lines are transformed into parallel planes and parallel lines. Of particular importance for our purpose is the case where the rank of matrix (2) is 1, so that the values of all its determinants of orders 3 and 2 vanish. The geometric meaning of this case is that the points of a certain plane s are left invariant, and that corresponding points lie on lines all parallel to a definite direction. Moreover, when P and P' are corresponding points and P_1 is the intersection of PP' with s, then $P'P_1 : PP_1 =$ constant. By a translation we can always make a_0, b_0, c_0 vanish, so that the origin $O \equiv O'$ becomes an invariant point. In this case the special affinity H, whose matrix is of rank 1, may always be written in the form

$$H \equiv \left\{ \begin{array}{l} x' = x + \lambda_1 (x + py + qz), \\ y' = y + \lambda_2 (x + py + qz), \\ z' = z + \lambda_3 (x + py + qz), \end{array} \right\} \tag{3}$$

where $x + py + qz = 0$ is the plane, all of whose points are invariant. Corresponding points lie on parallel lines whose direction is determined by the constant ratios:

$$(x' - x) / (z' - z) = \lambda_1 / \lambda_3, \quad (y' - y) / (z' - z) = \lambda_2 / \lambda_3.$$

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* *Pascal's Repertorium,* Vol. II (2d ed.), pp. 100–101.

The line joining any two distinct corresponding points $P'(x', y', z')$, $P(x, y, z)$ cuts s in a point P_1 so that

$$P'P_1/PP_1 = 1+\lambda_1+\lambda_2 p+\lambda_3 q = \text{const.},$$

as stated above. This constant is also equal to the value of the determinant

$$\Delta = \begin{vmatrix} 1+\lambda_1 & \lambda_1 p & \lambda_1 q \\ \lambda_2 & 1+\lambda_2 p & \lambda_2 q \\ \lambda_3 & \lambda_3 p & 1+\lambda_3 q \end{vmatrix}$$

of the substitution H. When $\Delta=0$, then the homologous affinity H becomes a parallel projection on the plane s. The affinity is, in this case, singular.

Through every point $P(x, y, z)$ of an affinity S there is just one system of three mutually orthogonal planes which is transformed into such an orthogonal system through $P'(x', y', z')$. If these two systems are chosen as coordinate planes, S assumes the simple form

$$D \equiv x' = ax, \quad y' = by, \quad z' = cz, \tag{4}$$

which is called a dilatation. As the two coordinate systems (which we may assume as having both the same sense) may be brought to coincidence by a rotation R we have the well-known

THEOREM I. *Every affinity S may be considered as the product of a rotation R and a dilatation D, so that $S=RD$.**

3. *Homologous Affinity.*

We shall now consider in particular the affinity of type H. Such an affinity is also determined by two tetrahedrons whose corresponding points lie on four non-coplanar parallel lines. The planes of corresponding faces, and of corresponding planes, in general, meet in lines of a fixed plane s, the plane of homology. The parallel lines joining corresponding points pass through the same infinite point, called center of homology. We shall call H an homologous affinity.

The question is, whether it is possible to represent S in the form $S=R_1D_1H$, in which the substitution

$$D_1 \equiv x' = \rho x, \quad y' = \rho y, \quad z' = \rho z, \tag{5}$$

is a similitude.

For this purpose we first prove

THEOREM II. *There always exist homologous affinities by which any ellipsoid is transformed into a sphere, and conversely.*

Let
$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \tag{6}$$

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* *Pascal's Repertorium, loc. cit.* Koenigs, "Leçon de cinématique" (1897), pp. 394–405.

be any ellipsoid, and assume $a>b>c$. The two systems of circular sections are parallel to the diametral planes

$$z=\pm\frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2-c^2}}\cdot x. \tag{7}$$

Consider the plane s, whose equation is obtained from (7) by choosing the — sign, and which cuts the ellipsoid in a circle. Through this as a great circle pass a sphere, whose equation will be

$$x^2+y^2+z^2=b^2. \tag{8}$$

It is easily shown that the ellipsoid (6) and the sphere (8) are inscribed in two right circular cylinders whose axes are in the xz-plane and have the slopes

$$m=\pm\sqrt{(b^2-c^2)/(a^2-b^2)}. \tag{9}$$

Denoting the coordinates of a point of the ellipsoid by x', y', z', and of a point on the sphere by x, y, z, and considering the cylinder obtained by taking the + sign in (9), it is found that the ellipsoid results from the sphere by the homologous affinity H, defined by

$$H^{-1}\equiv\left\{\begin{array}{l} x=x'-\dfrac{(ac+b^2)\sqrt{(a^2-b^2)(b^2-c^2)}}{ac\{a(b^2-c^2)+c(a^2-b^2)\}}\cdot\left\{c\sqrt{\dfrac{a^2-b^2}{b^2-c^2}}\cdot x'+az'\right\}, \\ y=y', \\ z=z'-\dfrac{(ac+b^2)(b^2-c^2)}{ac\{a(b^2-c^2)+c(a^2-b^2)\}}\cdot\left\{c\sqrt{\dfrac{a^2-b^2}{b^2-c^2}}\cdot x'+az'\right\}, \end{array}\right\} \tag{10}$$

with s as the invariant plane, and the direction in the xz-plane with the slope $m=+\sqrt{(b^2-c^2)/(a^2-b^2)}$ as that of the infinite center of homology. Two corresponding points P' and P of H are joined by a line cutting s in P_1, so that $P'P_1/PP_1=-\frac{ac}{b^2}=$ constant. The slope $m=-\sqrt{(b^2-c^2)/(a^2-b^2)}$ in (9) determines another homologous affinity with the same property, which is symmetrical with the first, with respect to the yz-plane. Now, an affinity transforms conjugate poles and polar planes and triplets of conjugate diameters of a quadric into corresponding poles and polar planes and triplets of conjugate diameters of the transformed quadric. Consequently, by the homologous affinity H^{-1} any three conjugate diameters of the ellipsoid (6) are transformed into three conjugate diameters of the sphere (8), which, as such, are orthogonal to each other. Likewise, any three conjugate radii OA', OB', OC' of the ellipsoid are transformed into three rectangular radii OA, OB, OC of the sphere. The lines AA', BB', CC' cut the sphere in three other points $A_1B_1C_1$, so that also OA_1, OB_1, OC_1 are orthogonal. The slope $-m$ determines two other

orthogonal trihedrals on the sphere, so that their extremities lie twice in sets on three parallel lines through A', B', C'.

But any three non-coplanar lines $A'A'_{-1}$, $B'B'_{-1}$, $C'C'_{-1}$, which bisect each other at O, as conjugate diameters uniquely determine an ellipsoid with O as a center. By Chasle's * or other † well-known methods the three rectangular-conjugate diametral planes, and the orthogonal-conjugate diameters, axes, may be constructed. Using these as coordinate axes, and denoting the semi-axes in the order of their magnitude by a, b, c, the equation of the ellipsoid may be written in the form (6). Then, by the method explained above we may construct the four orthogonal trihedrals on the corresponding affine sphere.

4. *A Certain Composition of Affinity.*

It is now possible to answer the question concerning the representation of a general affinity S in the form $S=R_1D_1H$.

According to Theorem I, let R be the rotation, and D the dilatation, so that $S=RD$ carries a point (x, y, z) into the point (x', y', z'). Around these points determine the corresponding orthogonal systems of coordinate axes, so that by S the sphere K, $x^2+y^2+z^2=1$, is transformed into the ellipsoid E, $x'^2/a^2+y'^2/b^2+z'^2/c^2=1$, with $a>b>c$. According to the method explained above construct the sphere K_2, so that E is obtained from K_2 by a homologous affinity H. Determine the orthogonal coordinate system G_2 through the center of K_2, corresponding to the orthogonal system through the center of E. The equation of K_2 with respect to G_2 is $x_2^2+y_2^2+z_2^2=b^2$. To G_2 apply the similitude

$$D_1^{-1}\equiv x_1=x_2/b, \quad y_1=y_2/b, \quad z_1=z_2/b. \tag{11}$$

Finally, by a definite rotation R_1^{-1} the system $G_1(x_1, y_1, z_1)$ is transformed into the original system $G(x, y, z)$. Conversely, by R_1 the sphere $x^2+y^2+z^2=1$ is transformed into the sphere K_1, $x_1^2+y_1^2+z_1^2=1$. By D_1, K_1 is transformed into the sphere K_2, $x_2^2+y_2^2+z_2^2=b^2$. By H, K_2 is transformed into the ellipsoid $x'^2/a^2+y'^2/b^2+z'^2/c^2=1$. With this the identity $S\equiv RD\equiv R_1D_1H$ is proved for non-singular affinities.

It is also true for a singular affinity S_s, in which all points (x, y, z) are transformed into points (x', y', z') which lie in a plane. If we choose this as the plane s, the dilatation D_s will have the form

$$D_s\equiv x'=ax, \quad y'=by, \quad z'=0z. \tag{12}$$

* Beck, *loc. cit.*

† Fiedler, *Darstellende Geometrie*, Vol. II (1885), pp. 329–330.

The homologous affinity H becomes a parallel projection upon a plane s, defined by

$$H_s \equiv x' = x - \frac{\sqrt{a^2 - b^2}}{b} \cdot z, \quad y' = y, \quad z' = 0, \tag{13}$$

as is easily found from (10), by solving for x', y', z', and letting lim $(c) = 0$. The ellipsoid becomes an infinitely thin disk E_s in the plane s, whose contour has the equation $x'^2/a^2 + y'^2/b^2 = 1$, and we find again $S_s = R_1 \cdot D_1 \cdot H_s$. Hence

THEOREM III. *Every general affinity in space is the product of a rotation, a similitude, and an homologous affinity. In case of a singular affinity, in which all points are transformed into points of a plane, the homologous affinity becomes a parallel projection upon this plane.*

5. *Pohlke's Theorem and its Generalization.*

Chasle's construction of the axes of an ellipsoid still holds when the three conjugate diameters $A'A'_{-1}$, $B'B'_{-1}$, $C'C'_{-1}$ are coplanar. As before, the three diametral planes cut the degenerate ellipsoid E_s in three ellipses $(A'B')$, $(B'C')$, $(C'A')$, which are inscribed in three parallelograms as shown in Fig. 1.

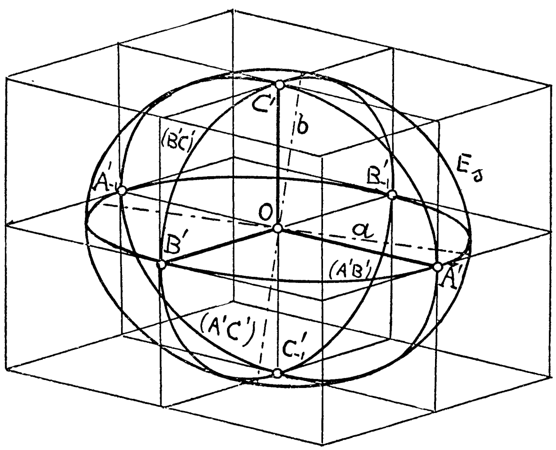

FIG. 1.

These ellipses are inscribed in an ellipse E_s in s, whose half-axes we denote by a and b $(a > b)$, and whose lines we choose as x'- and y'-axes, and the line through the center of E_s, perpendicular to s, as the z'-axis. Then the equation of E_s is precisely that given above as $x'^2/a^2 + y'^2/b^2 = 1$. The sphere K with the radius b, concentric with E_s, is now projected into E_s by a parallel projection H_s as defined by (13). Conversely, by the same formulas for H_s, and geometrically, it is easily verified, that to the ellipses $(A'B')$, $(B'C')$, $(C'A')$,

and their circumscribed parallelograms, correspond on K three great circles, whose planes are mutually perpendicular, and their circumscribed squares. To the complete rhombohedral lattice-work, circumscribed and inscribed to the ellipsoid as before, corresponds a cubical lattice-work connected in the same manner to the sphere, so that the rhombohedral is the parallel-projection (H_s)

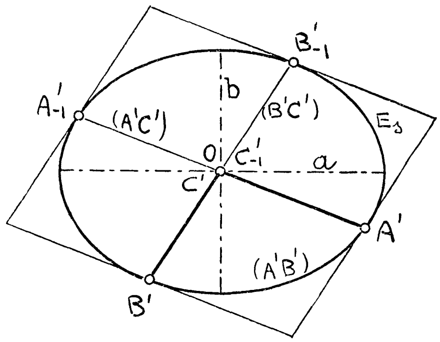

FIG. 2.

of the cubical lattice work. In this manner to the three distinct conjugate coplanar radii OA', OB', OC', correspond the three equal orthogonal radii OA, OB, OC. There are, in general, again four such sets of orthogonal radii.

The proposition is still true when one of the three coplanar radii, say OC', vanishes. The ellipse $(A'B')$ has $A'A'_{-1}$ and $B'B'_{-1}$ as conjugate diameters, while the ellipses $(B'C')$ and $(C'A')$ coincide with the segments $B'B'_{-1}$ and $A'A'_{-1}$, Fig. 2. The contour ellipse E_s coincides with the ellipse $(A'B')$. The

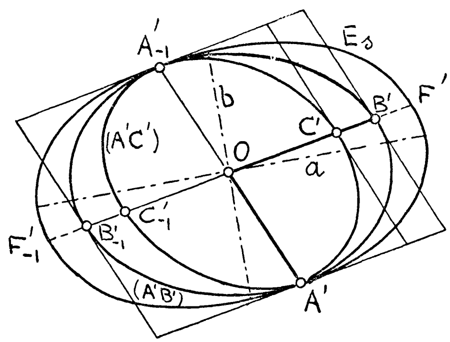

FIG. 3.

homologous sphere may be constructed precisely as before, so that to the ellipses $(A'B')$, ($B'C'=$ degenerate), ($C'A'=$ degenerate) correspond again three orthogonal great circles on the sphere K, and to the coplanar conjugate semi-diameters OA', OB', $OC'=0$, three orthogonal radiï of K.

When one of the angles, say the one between $B'B'_{-1}$ and $C'C'_{-1}$ is zero, Fig. 3, then the ellipses $(A'B')$, $(A'C')$ may be constructed as in the general

case. The ellipse $(B'C')$ degenerates into a straight line segment $F'F'_{-1}$, so that $|OF'|=|OF'_{-1}|=\sqrt{\overline{OB'}^2+\overline{OC'}^2}$. In this case E_s is the ellipse which has $A'A'_{-1}$ and $F'F'_{-1}$ as conjugate diameters, and which with the ellipses $(A'B')$, $(B'C')$, $(C'A')$, and the corresponding circumscribed parallelograms and conjugate diameters may be considered as the parallel projection H_s of a sphere K with three orthogonal great circles and the attached cubic lattice-work. In every case there exist equiaxial-orthogonal trihedrals ($OA=OB=OC$; $\sphericalangle AOB=\sphericalangle BOC=\sphericalangle COA=90°$) of which OA', OB', OC', whether coplanar or not, form a parallel projection. Hence, we may state Pohlke's Theorem in a generalized form.

THEOREM IV. *The vertex and the extremities of any three concurrent, coplanar or non-coplanar, straight line segments in space always lie in a definite order on four parallel lines through the vertex and the extremities of an equiaxial-orthogonal trihedral. In general, there are two distinct sets of four parallel lines each, and four sets of orthogonal trihedrals with this property. Not more than one segment, and not more than one angle between the segments of the given trihedral may vanish.*

As a general affinity S depends on twelve independent parameters, it is always possible to determine uniquely an affinity S in which any two proper tetrahedrons T' and T correspond to each other in a definite order. For example, $P'_1P'_2P'_3P'_4$ to $P_1P_2P_3P_4$. But we have proved that $S=R_1D_1H$, so that T' results from T by a rotation, followed by a similitude, and finally by an homologous affinity. When $P'_1P'_2P'_3P'_4$ are coplanar, then the substitution H becomes a parallel projection H_s, and S is a singular affinity, for which the determinant of the substitution vanishes. The result may be stated as

THEOREM V.* *If any two proper tetrahedrons $P'_1P'_2P'_3P'_4$ and $P_1P_2P_3P_4$ are given, it is always possible to determine a tetrahedron $P''_1P''_2P''_3P''_4$ similar (eventually congruent) to $P_1P_2P_3P_4$, so that the lines joining P'_1 and P''_1, P'_2 and P''_2, P'_3 and P''_3, P'_4 and P''_4 are parallel. This is still true when the points $P'_1P'_2P'_3P'_4$ form a proper plain quadrangle, or also when the segments $P'_4P'_1$, $P'_4P'_2$, $P'_4P'_3$ and the angles formed by them are subject to the necessary and sufficient conditions of Pohlke's Theorem.*

This theorem clearly contains Pohlke's and Reye's theorems as special cases.

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*The first part of this theorem concerning two proper tetrahedrons has also been proved by Hurwitz in a recent communication to the Swiss Mathematical Society, an abstract of which in *L'Enseignement Mathématique* reached the author several months after this paper was sent to the AMERICAN JOURNAL OF MATHEMATICS.

6. *Related Theorems.*

From the connection between the rhombohedral and cubical lattice-works discussed above, we deduce without difficulty

THEOREM VI. *A plain hexagon with three pairs of parallel, opposite sides, with the sides of each pair equal, may always be considered as the contour of a parallel projection of a cube. The net of six parallelograms constructed with each two adjacent sides of the hexagon as a pair of adjacent sides of a parallelogram, is the projection of the edges of the cube.*

Completing the rhombohedral lattice-works, determined by P_4P_1, P_4P_2, P_4P_3 and $P'_4P'_1$, $P'_4P'_2$, $P'_4P'_3$ as clinographic semi-axes of the rhombohedrons, and inscribing ellipsoids into these, with the clinographic axes in each case as triplets of conjugate diameters, we find

THEOREM VII. *If any two parallelopipeds (rhombohedrons) π' and π are given, it is always possible to find a parallelopiped π'' similar (eventually congruent) to π, so that corresponding vertices of π' and π'' lie on eight parallel lines (eventually counting multiplicities properly).*

In a similar manner we have

THEOREM VIII. *If any two ellipsoids E' and E are given, it is always possible to find an ellipsoid E'' similar (eventually congruent) to E, so that E' and E'' are inscribed to the same elliptic (circular) cylinder.*

Likewise as a special case of the foregoing,

THEOREM IX. *The contour of a parallel projection of any given ellipsoid upon a plane may be similar to any given ellipse.*

Finally,

THEOREM X. *It is always possible to circumscribe two (may be coincident) right circular cylinders to any ellipsoid.*